

Codeword Stabilized Quantum Codes for Asymmetric Channels

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Abstract—We discuss a method to adapt the codeword stabilized (CWS) quantum code framework to the problem of finding asymmetric quantum codes. We focus on the corresponding Pauli error models for amplitude damping noise and phase damping noise. In particular, we look at codes for Pauli error models that correct one or two amplitude damping errors. Applying local Clifford operations on graph states, we are able to exhaustively search for all possible codes up to length 9. With a similar method, we also look at codes for the Pauli error model that detect a single amplitude error and detect multiple phase damping errors. Many new codes with good parameters are found, including nonadditive codes and degenerate codes.

Index Terms—codeword stabilized quantum code, nonadditive code, asymmetric code, amplitude damping channel, phase damping channel

I. INTRODUCTION

Codeword stabilized (CWS) quantum codes constitute the by far most general systematic framework for constructing quantum error-correcting codes (QECC) [6], [7], [9]. It encompasses stabilizer codes [4], [5], [14], [32], as well as many nonadditive codes with good parameters [23], [28], [35]. Over the past years, it has been explored in various settings and has been applied in many different cases, leading to promising results [2], [15]–[17], [20], [25], [26], [33].

Most of the QECC constructed so far are for the depolarizing channel

$$\mathcal{E}_{\text{DP}}(\rho) = (1 - p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z), \quad (1)$$

where the Pauli X, Y, Z errors happen equally likely. (Here ρ denotes the density matrix representing the state of the quantum system.) The most general quantum channels allowed by quantum mechanics are completely positive, trace-preserving linear maps that can be represented in the Kraus decomposition $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ with $\sum_k E_k^\dagger E_k = I$ [27].

One example generalizing the depolarizing channel \mathcal{E}_{DP} is the asymmetric Pauli channel which sends ρ to

$$(1 - p_x - p_y - p_z)\rho + p_x X\rho X + p_y Y\rho Y + p_z Z\rho Z, \quad (2)$$

where the Pauli X, Y, Z errors happen with probabilities p_x, p_y, p_z , respectively [21]. Other asymmetric channels studied in the literature include the amplitude damping channel [8]

$$\mathcal{E}_{\text{AD}}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger, \quad (3)$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad (4)$$

for some damping parameter γ .

It has been demonstrated that designing QECC adaptively to specific error models can result in better codes [8], [11]–[13], [23], [24], [30], [31] and fault-tolerant protocols [1]. Although most of these codes are indeed CWS codes, there has been no systematic construction applying the CWS framework. In this work, we fill this gap by developing a method for finding CWS codes for asymmetric channels. Our method leads to many new codes with good parameters, including nonadditive codes and degenerate codes. These results demonstrate the power of the CWS framework for constructing good QECC.

II. ERROR MODELS

Depending on the noise model in different physical systems, we obtain different asymmetric quantum channels. We start with the amplitude damping channel \mathcal{E}_{AD} with Kraus operators given in Eq. (4), which models real physical processes such as spontaneous emission. If the system is at finite temperature, then the noise model will not only contain the Kraus operator A_1 corresponding to emission, but also A_1^\dagger corresponding to absorption [27]. Notice that

$$A_1 = \frac{\sqrt{\gamma}}{2}(X + iY), \quad A_1^\dagger = \frac{\sqrt{\gamma}}{2}(X - iY). \quad (5)$$

Hence, the linear span of the operators A_1 and A_1^\dagger equals the linear span of X and Y . We can then equivalently formulate the error model by using the Pauli operators X and Y , which happen with equal probability. That is, if a code is capable of correcting t X - and t Y -errors, it can also correct t A_1 - and t A_1^\dagger -errors.

Furthermore, notice that

$$A_0 = I - \frac{\gamma}{4}(I - Z) + O(\gamma^2), \quad (6)$$

while A_1 depends linearly on $\sqrt{\gamma}$. This then results in an asymmetry between the probabilities $p_x = p_y$ and p_z that the Pauli X, Y errors or the Pauli Z error, respectively, happen.

Apart from amplitude damping, another common noise in physical systems is dephasing, with Kraus operators given by $\sqrt{1-p}I$ and $\sqrt{p}(I \pm Z)/2$, or equivalently, in terms of I and Z with $p_z > 0$ and $p_x = p_y = 0$ [27]. In general, the system undergoes both amplitude damping and dephasing, resulting in a wide range for the parameters $p_x = p_y$ and p_z .

Therefore, in this work we consider the following asymmetric Pauli channel

$$\mathcal{E}_{\text{AS}}(\rho) = (1 - (2p_{xy} + p_z))\rho + p_{xy}(X\rho X + Y\rho Y) + p_z Z\rho Z, \quad (7)$$

where X and Y happen with equal probability $p_x = p_y = p_{xy}$.

In terms of Eq. (7), the asymmetric Pauli error model corresponding to amplitude damping is given by $p_{xy} \propto \gamma$ and $p_z \propto \gamma^2$. This is different from the amplitude damping error model in, e.g., [11], [18], [23], [24], [31], where the Kraus operators A_0 and A_1 are used. The main reason that we use the Pauli Kraus operators as our error sets is that this enables us to use the CWS framework to construct codes. Within the CWS framework, in order to transform the quantum error detection condition into a classical condition, it is more convenient to use Pauli errors, as we will discuss in Sec. III. In other words, since A_0 and A_1 are not Pauli operators, the CWS framework does not directly apply. Furthermore, due to Eq. (5) and Eq. (6), our error model does not only correct the errors A_0 and A_1 , but the resulting codes will be stronger in the sense that A_1^\dagger can be corrected as well.

In this work we consider three specific cases for asymmetric codes, as listed below. We use X_i, Y_i, Z_i to denote the Pauli X, Y, Z operators on the i th qubit. Notice that our method for generating the error sets is very general and can be straightforwardly generalized to deal with different relations between p_x, p_y , and p_z .

1. Codes correcting a single amplitude damping error: to improve the fidelity of the transmitted state from $1 - \gamma$ to $1 - \gamma^2$, one only needs to correct a single A_1 error and detect a single A_0 error [18]. In terms of Pauli operators, the corresponding error set is given by

$$\mathcal{E}^{\{1\}} = \{I\} \cup \{X_i, Y_i, Z_i, X_i X_j, Y_i Y_j, X_i Y_j, Y_i Y_j\}, \quad (8)$$

where $i, j = 1, \dots, n$. A code that detects this error set in fact also corrects a single A_1^\dagger error.

2. Codes correcting two amplitude damping errors: based on the analysis on the single error case above, the error set is given by

$$\mathcal{E}^{\{2\}} = \{E_\mu E_\nu : E_\mu, E_\nu \in \mathcal{E}^{\{1\}}\}. \quad (9)$$

A code that detects this error set in fact also corrects two A_1^\dagger errors.

3. Codes detecting both a single amplitude damping error and multiple dephasing errors: detecting $\{X_i, Y_i, Z_i\}$ suffices to detect an arbitrary single qubit error (including a single amplitude damping error), and detecting all

Z -errors up to weight r will allow to correct $\lfloor r/2 \rfloor$ Z -errors. The error set is

$$\mathcal{E}^{\{3\}} = \{I\} \cup \{X_i, Y_i : i = 1, \dots, n\} \cup \mathcal{Z}_r, \quad (10)$$

where \mathcal{Z}_r is the set of all Pauli Z operators up to weight r . A code that detects this error set in fact detects both an arbitrary error and r phase errors.

III. ALGORITHM TO SEARCH FOR CWS CODES

A QECC Q is a subspace of the space of n qubits (\mathbb{C}^2) $^{\otimes n}$ (here we focus on quantum systems of dimension $q = 2$, but the approach can be generalized to qudits of dimension $q > 2$). For a K -dimensional code space spanned by the orthonormal basis $\{|\psi_i\rangle : i = 1, \dots, K\}$ and an error set \mathcal{E} , there is a physical operation detecting all the elements $E_\mu \in \mathcal{E}$ (as well as their linear combinations) if the error detection condition [3], [22]

$$\langle \psi_i | E_\mu | \psi_j \rangle = c_\mu \delta_{ij}, \quad c_\mu \in \mathbb{C}, \quad (11)$$

is satisfied. The notation $((n, K))$ is used to denote a QECC with length n and dimension K .

Our goal is to find good codes detecting the error sets $\mathcal{E}^{\{j\}}$, for each of the three cases. For each code length n , we seek the largest dimension K of CWS codes for each error set $\mathcal{E}^{\{j\}}$, $j = 1, 2, 3$. This is done through a maximum clique search [7], by using the algorithms and programs developed in [34].

A. The CWS framework

An $((n, K))$ CWS code Q is described by two objects: 1) A stabilizer S that is an abelian subgroup of the n -qubit Pauli group, has order 2^n , and does not contain $-I$; the group S is called the word stabilizer. 2) A set of K n -qubit Pauli operators $W = \{w_\ell : \ell = 1, \dots, K\}$, which are called the word operators. There is a unique quantum state $|S\rangle$ stabilized by S , i.e., $s|S\rangle = |S\rangle$ for all $s \in S$. The code Q is then spanned by the basis vectors given by $|w_\ell\rangle = w_\ell|S\rangle$.

According to Eq. (11), the code Q detects the error set \mathcal{E} if and only if $\langle w_i | E | w_j \rangle = c_E \delta_{ij}$ for all $E \in \mathcal{E}$. When \mathcal{E} consists of Pauli matrices, this error-detecting condition can be written in terms of S and w_i as below [9]:

For all $E \in \mathcal{E}$,

$$\forall i \neq j : w_i^\dagger E w_j \notin \pm S \quad (12)$$

and

$$(\forall i : w_i^\dagger E w_i \notin \pm S) \quad \text{or} \quad (13)$$

$$(\forall i : w_i^\dagger E w_i \in S) \quad \text{or} \quad (14)$$

$$(\forall i : w_i^\dagger E w_i \in -S). \quad (15)$$

If condition (13) holds for all $E \in \mathcal{E}$ different from identity, then the code Q is nondegenerate, otherwise it is degenerate.

B. The CWS standard form

Every $((n, K))$ CWS code can be transformed, by local Clifford operations, into a standard form [9], where the word operators take the form $w_\ell = Z^{c_\ell}$ and the word stabilizer has generators of the form $S_i = X_i Z^{r_i}$, for some choices of classical n -bit strings c_ℓ and r_i . Here $Z^{c_\ell} = Z^{c_{\ell,1}} \otimes \dots \otimes Z^{c_{\ell,n}}$.

In the standard form, any n -qubit Pauli error, which can be written in the form $E = \pm Z^v X^u$ for some classical n -bit strings v and u , can be translated to classical errors via the map

$$\text{Cl}_S(E = \pm Z^v X^u) = v \oplus \bigoplus_{i=1}^n (u)_i r_i. \quad (16)$$

Now for the word operators $\{Z^{c_\ell} : c_\ell \in \mathcal{C}\}$, the error detection condition requires that the classical binary code \mathcal{C} detects all errors from $\text{Cl}_S(\mathcal{E})$, and that for each $E \in \mathcal{E}$

$$\text{Cl}_S(E) \neq \mathbf{0} \quad (17)$$

$$\text{or } \forall \ell: Z^{c_\ell} E = E Z^{c_\ell}. \quad (18)$$

If Eq. (17) holds for all $E \in \mathcal{E}$, the CWS code is nondegenerate, otherwise it is degenerate.

C. Local Clifford operations

To get to the standard form, one needs to apply local Clifford (LC) operations of the form $L = \bigotimes_i L_i$, where L_i are single-qubit Clifford operations [9]. This transforms the stabilizer S and word operators $\{w_\ell\}$ to the standard form, but at the same time also changes the error model.

For the depolarizing channel given in Eq. (2), the error set is invariant under LC operations, since in this model essentially all single-qubit errors happen equally likely. Therefore, in order to search for a CWS code, one can simply use the standard form by starting from a stabilizer of the form $S_i = X_i Z^{r_i}$, which corresponds to a graph state [19]. For a fixed length n , it is sufficient to consider all graph states up to LC equivalence as classified in [10]. This results in an exhaustive search for all possible CWS codes of length n .

Being able to restrict the search to graph states up to LC equivalence, instead of all stabilizer states of length n , has dramatically reduced the search space, and exhaustive search for single-error-correcting codes for the depolarizing channel up to length $n = 10$ has been carried out. It turned out that the best CWS code with length $n = 9$ has dimension $K = 12$, beating the best stabilizer code of dimension $2^3 = 8$ [35]; for $n = 10$ the best CWS code has dimension $K = 24$, again beating the best stabilizer code of dimension $2^4 = 16$ [20].

For the asymmetric channels as given in Eq. (7), however, considering only all graphs states as classified in [10] and the error sets $\mathcal{E}^{\{j\}}$ ($j = 1, 2, 3$) is not sufficient to exhaustively search for all possible CWS codes. This is due to the asymmetry between p_{xy} and p_z , which implies that the error sets are no longer invariant under LC operations. Therefore, in order to exhaustively search for all possible CWS codes by using the standard form, one will need to check all the possible error sets that are LC equivalent to a given $\mathcal{E}^{\{j\}}$.

Recall that the single-qubit Clifford group is generated by the Hadamard operator H and the phase operator P as given below [4], [14]

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \quad (19)$$

Since overall phase factors can be ignored, we only need to consider the action of the Clifford group on the Pauli matrices X, Y, Z modulo phase factors. The Clifford group acts as the permutation group S_3 on three letters (we use 1, 2, 3 to denote X, Y, Z , respectively). The group S_3 has order six, with the elements given by (in cycle notation) $\{\text{id}, (123), (132), (12), (13), (23)\}$, where id denotes the identity permutation. All error sets $\mathcal{E}^{\{j\}}$, $j = 1, 2, 3$, are invariant with respect to interchanging X and Y . Hence it is sufficient to consider one representative from each of the three right cosets of (12) , given by $\{\text{id}, (12)\}$, $\{(13), (132)\}$, and $\{(23), (123)\}$. So effectively, we only need to test, e.g., the three permutations $\{\text{id}, (13), (23)\}$.

Therefore, for each of the error sets $\mathcal{E}^{\{j\}}$, we have three cases for each qubit i : no permutation, permute Y and Z , or permute X and Z . To search for a length n code, this will reduce the total number of error sets from 6^n to 3^n for each graph state to be tested. Compared to codes for the depolarizing channel, the search space is enlarged by a factor of 3^n , due to the asymmetry between p_{xy} and p_z . Nevertheless, we can still handle the search for small n , in particular for the error sets $\mathcal{E}^{\{j\}}$, $j = 1, 2, 3$, up to length $n = 9$.

IV. RESULTS

As described above, in our search algorithm, we start from the CWS standard form and transform the error set $\mathcal{E}^{\{j\}}$ by LC operations. This has no effect on the code parameters $((n, K))$ found. However, to present the CWS codes found, we fix the error set $\mathcal{E}^{\{j\}}$ and equivalently transform the CWS standard form into a general CWS code.

A. Codes correcting a single amplitude damping error

We have conducted an exhaustive search for the error set $\mathcal{E}^{\{1\}}$ up to length $n = 9$, resulting in CWS codes correcting a single amplitude damping error. As already mentioned, the codes found can not only correct a single error given by the Kraus operator A_1 , at the same time they also correct a single error A_1^\dagger . In other words, the codes correct both single X errors and single Y errors (and detect single Z errors as well). We summarize our results in Table I.

As we can see from the table, for the lengths 6, 7, 8, 9, our codes outperform the best single-error-correcting codes for the depolarizing channel—which also correct the error set $\mathcal{E}^{\{1\}}$, i.e., a single amplitude damping error. In particular, for lengths 8 and 9, the best CWS codes we have found (of dimensions 10 and 20 respectively) are nonadditive codes.

For lengths 6, 8, 9, our codes also outperform the best known CSS codes that are specifically designed to detect the error set $\mathcal{E}^{\{1\}}$, based on a construction proposed in [14] (see also [31]). Therefore, for these lengths, we have found good

TABLE I

Dimension K of CWS codes $((n, K))$ of length n detecting the error set $\mathcal{E}^{\{1\}}$ for different length n . The column $d = 3$ lists the largest dimension of CWS codes that correct a single error for the depolarizing channel. The column $\mathcal{E}^{\{1\}}$ lists the largest dimension of CWS codes found detecting the error set $\mathcal{E}^{\{1\}}$. The column CSS lists the largest dimension of the known Calderbank-Shor-Steane (CSS) codes [4], [32] that can correct the error set $\mathcal{E}^{\{1\}}$, based on a construction proposed in [14]. The column GF(3) lists the largest dimension of codes correcting a single amplitude damping error based on a construction proposed in [31].

n	$d = 3$ [4], [35]	$\mathcal{E}^{\{1\}}$	CSS [14]	GF(3) [31]
5	2	2	2	2
6	2	4	2	5
7	2	8	8	8
8	8	10	8	16
9	12	20	16	24

codes that outperform all the previously known constructions for detecting the error set $\mathcal{E}^{\{1\}}$.

Notice that the existence of a CWS code with dimension $K = 4$, and hence a subcode of dimension 3, implies the existence of a stabilizer code with the same parameters [7, Theorem 7]. Hence the $((6, 4))$ codes we found, as listed in Table I, include stabilizer codes encoding two qubits. As an example, one such code has stabilizer S generated by

$$\begin{aligned} X & X & I & I & Z & Z \\ X & Z & I & Z & I & X \\ Z & I & Y & Z & Y & Z \\ I & I & Z & X & I & Z \end{aligned}$$

It is straightforward to check that this code detects the error set $\mathcal{E}^{\{1\}}$, since no elements in $\mathcal{E}^{\{1\}}$ is also in $C(S) \setminus S$, where $C(S)$ is the centralizer of the stabilizer S .

However, with the exception of $n = 7$, the single-error-correcting codes constructed in [31] have larger dimensions than our codes. The codes constructed in [31] are specifically designed to correct the Kraus operators A_0 and A_1 , these codes cannot detect the error set $\mathcal{E}^{\{1\}}$. As detection of the errors $\mathcal{E}^{\{1\}}$ implies that a single error A_1^\dagger can be corrected as well, it is not a surprise that our codes have smaller dimensions.

Notice that the codes constructed in [31] are also CWS codes, but errors are handled in a different way than the Pauli error set $\mathcal{E}^{\{1\}}$. It remains open how to generalize the method of [31] to deal with more than one amplitude damping error, while the error set $\mathcal{E}^{\{1\}}$ can naturally be generalized, e.g., to $\mathcal{E}^{\{2\}}$ for correcting two amplitude damping errors, as demonstrated next.

B. Codes correcting two amplitude damping errors

We have performed an exhaustive search for codes correcting two amplitude damping errors, i.e., detecting the error set $\mathcal{E}^{\{2\}}$, up to length $n = 9$. In fact, the resulting codes correct any combination of X and Z errors up to weight two, as well as a single Z error.

No non-trivial CWS codes are found for length $n \leq 8$. For length $n = 9$, two LC-inequivalent codes encoding a single qubit have been found. These are both stabilizer codes

encoding a single qubit, since the corresponding classical code \mathcal{C} is trivially linear [9].

One code has the stabilizer S_1 generated by

$$\begin{aligned} X & I & I & I & I & I & I & I & Z \\ Z & I & I & I & X & Z & I & Z & X \\ I & X & I & I & I & I & I & Z & I \\ I & Z & I & I & X & I & Z & X & Z \\ I & I & X & I & I & Z & I & I & I \\ I & I & Z & I & Y & Y & Z & Z & I \\ I & I & I & X & I & I & Z & I & I \\ I & I & I & Z & Y & Z & Y & I & Z \end{aligned}$$

The other code has the stabilizer S_2 generated by

$$\begin{aligned} X & I & I & I & I & I & I & I & Z \\ Z & I & I & I & Z & Z & Z & I & X \\ I & X & I & I & I & I & I & Z & I \\ I & Z & I & Z & Z & Y & I & Y & Z \\ I & I & X & I & I & I & Z & I & I \\ I & I & Z & Z & Z & X & X & Z & I \\ I & I & I & X & I & Z & I & I & I \\ I & I & I & I & X & I & Z & Z & Z \end{aligned}$$

It is straightforward to check that these codes detect the error set $\mathcal{E}^{\{2\}}$, since no elements in $\mathcal{E}^{\{2\}}$ is also in $C(S_i) \setminus S_i$ (for $i = 1, 2$). Furthermore, both codes are degenerate since some of the elements in $\mathcal{E}^{\{2\}}$ are indeed in S_i , for instance $X_1 Z_9$.

These codes outperform the $((10, 2))$ code found in [11]. Recall that Shor's nine-qubit code, having the same parameters $((9, 2))$ as our codes, also corrects two amplitude damping errors [14]. However, Shor's code only corrects the Kraus operators A_0 and A_1 , but does not detect the error set $\mathcal{E}^{\{2\}}$. Therefore, for length $n = 9$, we have found good codes that outperform all the previous known constructions for detecting the error set $\mathcal{E}^{\{2\}}$.

C. Codes detecting a single amplitude damping error and detecting multiple dephasing errors

For the error set $\mathcal{E}^{\{3\}}$, we have performed an exhaustive search for different lengths n and Z -error detecting capabilities r up to $n = 8$, and a random search starting from randomly selected graph states for $n = 9$ and different r . Our results are listed in Table II. We compare our results with the best stabilizer codes that detect all errors up to weight r as given in [4], and the codes detecting a single amplitude damping errors and Z errors up to weight r as found in [12].

As we can see from the table, for most lengths n and Z -error weight r , the CWS codes found outperform the known results. The entries for which we did not find improvements are $n = 6$, $r = 1$ and $n = 8$, $r = 1$. Codes with $r = 1$ detect single Pauli errors, i.e., they are codes of minimum distance two. For even length, the corresponding stabilizer codes are known to have the largest possible dimension for single-error-detecting codes [29]. For odd length $n = 5, 7, 9$, we find codes with parameters matching those of the code family $((2m + 1, 3 \times 2^{2m-3}, 2))$ given in [29]. Whenever the dimension is a power of two, the codes we found include stabilizer codes.

TABLE II

Dimension K of CWS codes detecting the error set $\mathcal{E}^{\{3\}}$ for different length n and parameter r . For each value of r , the first column lists the largest dimension of stabilizer codes that detect all errors up to weight r as given in [4]; the second column lists the largest dimension of asymmetric codes detecting a single amplitude damping error and phase errors up to weight r as found in [12]; the third column lists the largest dimension of the CWS codes found by our search for codes detecting the error set $\mathcal{E}^{\{3\}}$. ‘—’ means that no non-trivial codes exist based on the construction. The numbers labeled with * are the best parameters found by random search; otherwise the maximal dimension is obtained by exhaustive search.

n/r	1			2			3			4			5			6			7		
	stab.	[12]	CWS	stab.	[12]	CWS	stab.	[12]	CWS	stab.	[12]	CWS	stab.	[12]	CWS	stab.	[12]	CWS	stab.	[12]	CWS
5	4	5	6	2	—	4	—	—	2	—	—	2	—	—	—	—	—	—	—	—	—
6	16	16	16	2	2	8	—	—	4	—	—	2	—	—	2	—	—	—	—	—	—
7	16	22	24	2	8	16	—	—	8	—	—	2	—	—	2	—	—	2	—	—	—
8	64	64	64	8	8	20	—	8	16	—	—	4	—	—	2	—	—	2	—	—	2
9	64	93	96*	8	16	40*	—	8	20*	—	—	6*	—	—	4*	—	—	2*	—	—	2*

These results demonstrate the power of the CWS framework for constructing good QECC, even with random search.

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